Decoupling Estimate and Its Applications in Schrödinger Equations

去耦合不等式及其在薛定谔方程中的应用

Zhaozhe Liu (刘兆哲)

Topics:

- Set-up of the Carleson Problem; Previous results
- Sequential Carleson Problem
- Two negative results of the sequential Carleson Problem

We discuss the following Cauchy problem of a free Schrödinger equation,

$$\begin{cases} iu_t - \Delta_x u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where $f \in H^s(\mathbb{R}^n)$ with $\|f\|_{H^s(\mathbb{R}^n)} = \left(\int (1+|\xi|^2)^s |\widehat{f(\xi)}|^2 \, d\xi \right)^{\frac{1}{2}}$.
Denote the solution as $u(x,t) := e^{it\Delta} f(x)$, where

$$e^{it\Delta} f(x) = (2\pi)^{-\frac{n}{2}} \int e^{i\left(x\cdot\xi + t\cdot|\xi|^2\right)} \widehat{f}(\xi) \, d\xi.$$

Carleson Problem: When the almost convergence property, i.e.

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \textbf{a.e.}$$

for $f \in H^s(\mathbb{R}^n)$ holds?

Careleson, 1980	True for $f \in H^{\frac{1}{4}}(\mathbb{R})$, i.e. when $n = 1$, $s \geq \frac{1}{4}$.
Dahlberg and Kenig, 1982	$n = 1$, $s \geq \frac{1}{4}$ is sharp.
Sjölin and Vega, 1985	$s > \frac{1}{2}, \forall n$.
Bourgain, 1995	Improved to $f \in H^{\frac{1}{2}-\epsilon}(\mathbb{R}^2)$ when $n = 2$. Later improved by Tao, Vargas, et al.
Bourgain 2012	For $n \geq 3$, $s > \frac{1}{2} - \frac{1}{4n}$ is a sufficient condition by mutltilinear estimates for extension operators. For $n \geq 4$, a necessary condition is $s \geq \frac{1}{2} - \frac{1}{2n}$.
Bourgain 2016	For all $s < \frac{n}{2(n+1)}$, there exists counter-example.
Du-Guth-Li, 2017	When $n = 2$, convergence for $s > \frac{1}{3}$, i.e. $s > \frac{n}{2(n+1)}$ is sharp when $n = 2$.
Du-Guth-Li-Zhang, 2018	When $n \geq 3$, $s > \frac{n+1}{2(n+2)}$ by linear refined Strichartz
Du-Zhang, 2019	For $n \geq 3$, $s > \frac{n}{2(n+1)}$ by a broad-narrow analysis, multilinear refined Strichartz estimate and decoupling.

Theorem (Du and Zhang 2019)

For every $f \in H^s(\mathbb{R}^n), n \geq 3$ *with* $s > \frac{n}{2(n+1)}$,

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x) \text{ almost everywhere.}$$

To show the above, it suffices to show the following maximal estimate[1],

$$\| \sup_{0<t\leq 1} |e^{it\Delta} f| \|_{L^2(B^n(0,1))} \lesssim_s \|f\|_{H^s(\mathbb{R}^n)}.$$

Remark

The reduction from the maximal estimate to the convergence is due to the ***Nikisin-Stein*** *maximal principle. Hickman 2023*

[1] Du and Zhang 2019, "Sharp L_2 Estimates of the Schrödinger Maximal Function in Higher Dimensions", *Annals of Mathematics*.

A Follow-Up Question: When does $\lim_{t\to 0} e^{it\Delta} f(x) = f(x)$, $\mu - a.e.$ where μ is an α-dimensional measure.

That is to say the set on which the divergence fails is of Hausdorff dimension α.

Denote the size of the divergence set,

$$\alpha_n(s) := \sup_{f \in H^s(\mathbb{R}^n)} \dim_H \{x \in \mathbb{R}^n : \lim_{t\to 0} e^{it\Delta} f(x) \neq f(x)\}.$$

where the Hausdorff dimension is defined via the Hausdorff content, for a Borel set $A \subseteq \mathbb{R}^n$

$$H^\alpha_\delta(A) := \inf_{\text{all cover of radius } < \delta} \{\sum \text{diam}(B_i)^\alpha : A \subseteq \cup_i B_i\}.$$

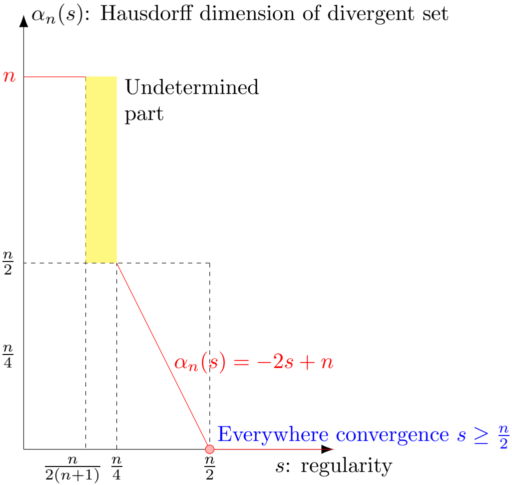


Figure: Previous results on $\alpha_n(s)$

Variants of Carleson Problem

Consider

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x), \quad a.e. x, \forall f \in H^s.$$

where $\{t_n\}$ is in a Lorentz space $\ell^{r,\infty}(\mathbb{N})$, for $0 < r < \infty$,

$$\{t_n\} \in \ell^{r,\infty} \Longleftrightarrow \sup_{b>0} b^r \#\{n \in \mathbb{N} : |t_n| > b\} < \infty.$$

Again, we could reduce the proof of the above convergence to the following maximal estimate.

$$\| \sup_{t_n} |e^{it_n\Delta} f| \|_{L^2(B(0,1))} \lesssim \|f\|_{H^s}.$$

Sequential Case, $n = 1$ and $n \geq 2$

Proposition (Dimou and Seeger 2020)

Let $n = 1$. Then $e^{it_n\Delta} f \to f, a.e.$ ***if and only if*** *$s \geq \min\{\frac{r}{2r+1}, \frac{1}{4}\}$.*

Theorem (Cho, Ko, et al. 2023, Li, Wang, and Yan 2023)

Let $n \geq 2$ and $r \in (0, \infty)$. For any decreasing sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$, $t_n \to 0$, the following maximal estimate holds for any $s > \min\{\frac{r}{\frac{n+1}{n}r+1}, \frac{n}{2(n+1)}\}$, and $f \in H^s(\mathbb{R}^n)$,

$$\lim_{n\to\infty} e^{it_n\Delta} f(x) = f(x), \quad a.e. x \in \mathbb{R}^n.$$

Sequential Case, $n \geq 2$

Theorem (Li, Wang, and Yan 2023 (Negative))

For each $r \in (0, \infty)$, there exists a sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N})$, the corresponding maximal estimates fails if $s < s_0 := \min\{\frac{r}{\frac{n+1}{n}r+1}, \frac{n}{2(n+1)}\}$.

Extend previous negative results to fractal settings by following Li-Wang-Yan's approach.

- Construction 1 (originally proposed by Luca and Rogers). Construction of divergence set $\Rightarrow$ Initial Datum.
- Construction 2 (originally proposed by Luca and Ponce). Construction of initial datum $\Rightarrow$ Divergence Set. (Easier to handle)

Theorem (Main Result 1)

Let $\frac{(3n+1)}{4} \leq \alpha \leq n$*. Then for any*

$$s < \min\{\frac{(n-1)(1-r)+1}{4}, \frac{1}{2}(n-\alpha) + \frac{(2\alpha - n)}{2(n+1)}\},$$

there exists a sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N}), r > 0$ *with* $t_n \to 0$ *when* $n \to \infty$*, and an initial datum* $u_0 \in H^s(\mathbb{R}^n)$ *such that*

$$\limsup_{t_n \to 0} |u(x,t)| = \infty,$$

for all x *in a set of positive* α*-dimensional measure.*

The initial function is defined via its (1)-dim part and $(n-1)$-dim part:

$$u_0(x) := \sum_{j\in\mathbb{N}} e^{i\pi\lambda^j(1,\theta_j)\cdot x}\phi(\lambda^{\frac{j}{2}}x_1)g_j(\overline{x})$$
$$= \sum_{j\in\mathbb{N}}(e^{i\pi\lambda^j x_1}\phi(\lambda^{\frac{j}{2}}x_1))(e^{i\pi\lambda^j\theta_j\cdot\overline{x}}g_j(\overline{x})),$$

where $\widehat{\phi} := \chi_{(-\epsilon_1,\epsilon_1)}$, $\widehat{g_j} := \lambda^{j\delta}|\Omega^j|^{-1}\chi_{\Omega_j}, 0<\delta<\frac{\sigma}{4}$ and $\theta_j \in \mathbb{S}^{n-2}$ being a direction.

$$\Omega^j = \{\overline{\xi} \in 2\pi\lambda^{j(1-\sigma)}\mathbb{Z}^{n-1} : \lambda^j \le |\xi_m| \le \lambda^{j+1}, m=2,\dots,n\}$$
$$+ Q(0, \frac{\epsilon_1}{\sqrt{n-1}}), j\in\mathbb{N}.$$ $\epsilon_1 > 0$ is a small constant.

A set upon which $|e^{it\Delta} f_{\theta_j}|$ is large. To do this, we may consider a "dual" space of Ω^j where the phase $\overline{x} \cdot \overline{\xi} + t|\overline{\xi}|^2 \sim 2\pi\mathbb{Z}^{n-1}$.

$$\text{Space } X^j_{t\theta_j} = \{\overline{x} \in \lambda^{j(\sigma-1)}\mathbb{Z}^{n-1} : |\overline{x}| \leq 2\} + \mathring{Q}(t\theta_j, \epsilon_2\lambda^{-j}),$$

$$\text{Time } T^j_{x_1} = \{t \in \lambda^{j(2\sigma-1)}\mathbb{Z} : x_1 < t < x_1 + \lambda^{-\frac{j}{2}}\}.$$

$$\Gamma^j_{t\theta_j} := X^j_{t\theta_j} \setminus \cup_{j<k\leq 2j} X^{k,\delta}_{\lambda^{k-j}t\theta_k},$$

$$X^{k,\delta}_{\lambda^{k-j}t\theta_k} := \{\overline{x} \in \lambda^{k(\sigma-1)}\mathbb{Z}^{n-1} : |\overline{x}| \leq 2\} + Q(\lambda^{k-j}t\theta_k, \epsilon_2\lambda^{-k(1-2\delta)}).$$

$$\Gamma_{x_1^j} := \bigcup_{t\in T^j_{x_1}} \Gamma^j_{t\theta_j}, \text{ and } \Gamma^j := \{x \in \mathbb{R}^n : x_1 \in (0, \frac{1}{2}), \overline{x} \in \Gamma^j_{x_1}\}.$$

$$\Gamma = \limsup_k \Gamma^k = \cap_{j\geq 1} \cup_{k\geq j} \Gamma^k$$

In order to have $u_0(x) \in H^s(\mathbb{R}^n)$, in other words,

$$\|u_0\|_{H^s(\mathbb{R}^n)} \sim \left(\sum_j \lambda^{2js} \|\chi_{A(\lambda^j)} \widehat{u_0}\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}} + \|u_0\|_{L^2(\mathbb{R}^n)} < \infty.$$

The Plancherel Theorem implies that

$$\begin{aligned} \lambda^{2js} \|\chi_{A(\lambda^j)} \widehat{u_0}\|^2_{L^2(\mathbb{R}^n)} &= \lambda^{2js} \|(e^{i\pi\lambda^j x_1} \phi(\lambda^{\frac{j}{2}} x_1))(e^{i\pi\lambda^j \theta_j \cdot \overline{x}} g_j(\overline{x}))\|^2_{L^2(\mathbb{R}^n)} \\ &\sim \lambda^{2js} \cdot \lambda^{-\frac{j}{2} + 2j\delta + (1-n)j\cdot\sigma} \\ &\sim \lambda^{j(2s - \frac{1}{2} + 2\delta + (1-n\sigma))}. \end{aligned}$$

Since $\lambda > 1$ and $\sum_j \lambda^{2js} \|\chi_{A(\lambda^j)} \widehat{f}\|^2_{L^2(\mathbb{R}^n)} < \infty$, the exponent $2s - \frac{1}{2} + 2\delta + (1-n)\sigma < 0$, therefore it suffices to have

$$s < \frac{(n-1)\sigma}{2} + \frac{1}{4} - \delta.$$

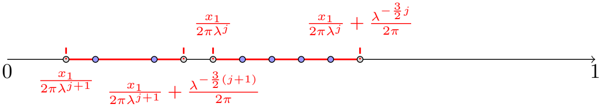
Sequential Structure

Time sequence $\{\frac{t}{2\pi\lambda_j}\}$ is taken from
$T^j_{x_1} = \{t \in \lambda^{j(2\sigma-1)}\mathbb{Z} : x_1 < t < x_1 + \lambda^{-\frac{j}{2}}\}$ where

$$\frac{t_j}{2\pi\lambda^j} \in \{\frac{\lambda^{j(2\sigma-2)}}{2\pi}\mathbb{Z} : \frac{x_1}{2\pi\lambda^j} < \frac{t}{2\pi\lambda^j} < \frac{x_1}{2\pi\lambda^j} + \frac{\lambda^{-\frac{3}{2}j}}{2\pi}\}.$$

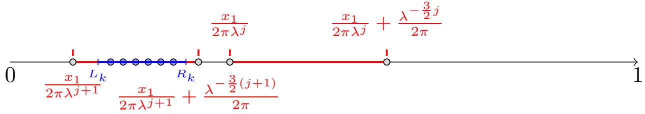
When $\frac{t_j}{2\pi\lambda^j} \to 0$, we have for $x \in \Gamma$,

$$|u(x, \frac{t_j}{2\pi\lambda^j})| \to \infty.$$



Lemma

Given $\{t_n\} \subseteq [0,1]$ *a sequence of positive numbers,* $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ *where* $0 < r < \infty$ *if there exists an uniform upper bound for any* $b > 0$*, such that* $\sup_{b>0} b^r \#\{t_n : b < t_n \leq 2b\} < A < \infty$*.*



$$
\begin{aligned}
b^r \cdot \#\{b < \frac{t_{j(k)}}{2\pi\lambda^{j(k)}} \leq 2b\} &\lesssim b^{r+1} \cdot \lambda^{j(2-2\sigma)} \\
&\lesssim \lambda^{(r+1)(1-j)} \cdot \lambda^{j(2-2\sigma)} \\
&= \lambda^{j(1-2\sigma-r)} \cdot \lambda^{(r+1)} < \infty.
\end{aligned}
$$

Now we have that the former lemma holds when $1 - 2\sigma - r \leq 0 \Rightarrow r \geq 1 - 2\sigma$.

Theorem (Lucà and Ponce-Vanegas 2022)

Let $n \geq 2$, and $\frac{n}{2} < \alpha \leq d$. Then, for any

$$s < \frac{n}{2(n+2)}(n+1-\alpha),$$

there ***exists*** *$u_0 \in H^s(\mathbb{R}^n)$ such that*

$$\limsup_{t \to 0} |u(x,t)| = \infty,$$

for all x in a set of positive $\alpha-$Hausdorff measure.

Theorem (Main Result 2)

Let $\frac{n}{2} < \alpha \leq n, n \geq 2$*, then for any*

$$s < \min\{\frac{n^2}{2(n+1)} - \frac{(n-1)(r+1)}{4(n+1)}, \frac{1}{2}(n-\alpha) + \frac{(2\alpha - n)}{2(n+1)}\},$$

there exists a sequence $\{t_n\}_{n=1}^{\infty} \in \ell^{r,\infty}(\mathbb{N}), r > 0$ *with* $t_n \to 0$ *when* $n \to \infty$*, and an initial datum* $u_0 \in H^s(\mathbb{R}^n)$ *such that*

$$\limsup_{t_n \to 0} |u(x,t)| = \infty$$

for all x *in a set of positive* α*-dimensional measure.*

Define the initial datum by its 1-dim part and the $(n-1)$-dim part.

$$f_{D_k} = f_1(x_1)\tilde{f}(\tilde{x})$$

where

$$f_1(x_1) := e^{2\pi i R_k x_1}\varphi(R_k^{\frac{1}{2}} x_1),$$

$\operatorname{supp}\hat{\varphi} \subseteq B(0,1)$ and $\varphi(0) = 1$.

$$\tilde{f}(\tilde{x}) := \prod_{j=2}^{n} \varphi(x_j) \left(\sum_{\frac{R_k}{2D_k} < \ell_j < \frac{R_k}{D_k}} e^{2\pi i D_k \ell_j x_j} \right),$$

Theorem

Given a constant $c \ll 1$ *and an integer* $q > 0$ *such that* $\frac{R_k}{D_k q} \gg \sqrt{\ln q}$. *If* f_{D_k} *is an initial datum given as above, then we have*

$$\frac{|e^{it\Delta} f_{D_k}(x)|}{\|f_{D_k}\|_2} \gtrsim R_k^{\frac{1}{4}} \left(\frac{R_k}{D_k q} \right)^{\frac{n-1}{2}},$$

for (x, t) *such that* $0 < t \in \mathcal{N}_{d_k}(\frac{2p_1}{D_k^2 q})$ *where* $d_k := \frac{c}{R_k^2}, t \ll \frac{1}{R_k}$ *and*

$$x \in E_{q,D_k} \cap [0, c]^n.$$

E_{q,D_k} *is defined by*

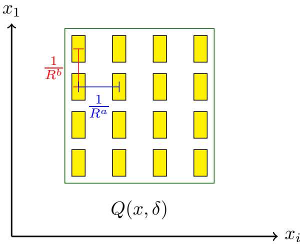
$$\begin{cases} x_1 \in 2\frac{p_1 R_k}{q D_k^2} + [-cR_k^{-\frac{1}{2}}, cR_k^{-\frac{1}{2}}], \\ x_j \in \frac{p_j}{q D_k} + [-cR_k^{-1}, cR_k^{-1}], j = 2, \ldots, n, \end{cases}$$

where $(\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_n}{q})$ *is an admissible fraction.*

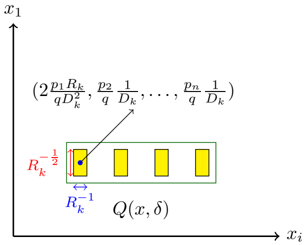
$\sup_{0<t<1} |e^{it\Delta} f_{D_k}|$ is, in fact, large on the set

$$\bigcup_{1 \le q \le Q_k} E_{q,D_k} \cap \left([\frac{c}{10}, c] \times [0,c]^{n-1} \right),$$

when $0 < c \ll 1$ and $\frac{R_k}{D_k Q_k} \gg \sqrt{\ln Q_k}$.



Definition of a, b



Geometry of a single slab s

$$R_k^{-a} := \frac{1}{Q_k^{\frac{n}{n-1}} D_k} \geq R_k^{-1} \Rightarrow a \leq 1. \quad R_k^{-b} := \frac{R_k}{Q_k D_k^2} \geq R_k^{-\frac{1}{2}}, \Rightarrow b \leq \frac{1}{2}.$$

$$Q_k = R_k^{\frac{n-1}{n+1}(2a-b-1)}, \quad D_k = R_k^{\frac{n-(n-1)a+nb}{n+1}}.$$

Define the initial datum $g_{a,b}$ as the following

$$g_{a,b}(x) := \sum_{k'=4k, k \geq k_0} \frac{k'}{R_{k'}^s} \frac{f_{D_{k'}}(x)}{\|f_{D_{k'}}\|_2}.$$

Define $F_k := \bigcup_{s \in \mathcal{A}_k} s$ and

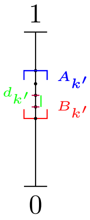
$$F := \limsup_{k \to \infty} F_k = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} F_k.$$

We eventually show that the Hausdorff dimension of divergent set $\tilde{F} \cap \left([\frac{c}{10}, c] \times [0, c]^{n-1}\right)$ is $\alpha := \frac{1}{2} + (n-1)a + b$, i.e. $\dim_H \left(\tilde{F} \cap \left([\frac{c}{10}, c] \times [0, c]^{n-1}\right)\right) = \alpha$.

$|e^{it\Delta} g_{a,b}(x)|$ is large at time $t_{\text{divergent}} \sim \frac{1}{R^{k'}}$. Construct an interval $[B_{k'}, A_{k'}] \subseteq [0,1]$ with length $\simeq \frac{1}{R_{k'}}$ at each frequency,

$$[B_{k'}, A_{k'}] := [\frac{1}{2R_{k'}}, \frac{2}{R_{k'}}], k' = 4k, k = 1, 2, \dots,$$

Then we have $B_{k'} = \frac{1}{2R_{4k}} = \frac{1}{2 \cdot 2^{4k}} = \frac{1}{2^{4k+1}}$ and $A_{k'+4} = \frac{2}{R_{4(k+1)}} = \frac{1}{2^{4k+3}} = \frac{1}{4} B_{k'} < \frac{1}{2} B_{k'}$.



$|e^{it\Delta} f_{D_{k'}}|$ is large when $t \in \mathcal{N}_{d_{k'}}(\frac{2p_1}{D_{k'}^2} q)$ whenever $d_{k'} \sim \frac{c}{R_{k'}^2}$. Now let us divide each $[B_{k'}, A_{k'}]$ by $d_{k'} := \frac{1}{R_{4k}^{\sigma}}$ with $\sigma > 2$.

$$
\begin{aligned}
b^r \#\{b < t_n < 2b\} &= b^r \cdot \frac{b}{\frac{1}{R_{4k}^{\sigma}}} \\
&= b^{r+1} R_{4k}^{\sigma} \\
&\lesssim (R_{4k})^{-(r+1)+\sigma} < \infty, \text{ as } b < A_{k'} = \frac{2}{R_{4k}},
\end{aligned}
$$

once we have $r \geq -1 + \sigma$. choose $\sigma := 2\alpha > 2 \cdot \frac{n}{2} = n \geq 2$, then we have a sequence of divergent time with $\{t_n\} \in \ell^{-1+2\alpha,\infty}(\mathbb{N})$.

Recent Progress by Cho and Eceizabarrena

Theorem (Cho and Eceizabarrena 2024, March)

Let $n \geq 2$*.*

- *If* $n-1 \leq \alpha \leq n$*,*

$$s_c^{\max} \geq \frac{n-\alpha}{2} + \min(\frac{2\alpha - n}{2(n+1)}, \frac{r(2\alpha - n)}{r(n+1) + 2\alpha - n}).$$

- *If* $\frac{n}{2} < \alpha \leq n-1$

$$s_c^{\max} \geq \frac{n-\alpha}{2} + \min(\frac{2\alpha - n}{2(n+1)}, \frac{r\alpha}{r(n+1) + n}).$$

Key Observation:

- $\hat{f}_1(\xi) = \frac{1}{R_k^{\frac{1}{2}}} \widehat{\varphi}(\frac{\xi - R_k}{R_k^{\frac{1}{2}}}) \Rightarrow \frac{1}{S}\widehat{\varphi}(\frac{\xi - R}{S})$.
- $t \lesssim \min\left(\frac{1}{S^2}, \frac{1}{R}\right)$. The authors focus on the case where $t \lesssim \frac{1}{S^2} \ll \frac{1}{R}$, i.e. $R^{1/2} \ll S \leq R$. We can then solve S.